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ORTHOGONAL POLYNOMIALS WITH WEIGHT FUNCTION $(1-x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1)$

BY

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ABSTRACT. We study orthogonal polynomials for which the weight function is a linear combination of the Jacobi weight function and two delta functions at 1 and -1. These polynomials can be expressed as ${}_4F_3$ hypergeometric functions and they satisfy second order differential equations. They include Krall's Jacobi type polynomials as special cases. The fourth order differential equation for the latter polynomials is derived in a more simple way.

0. **Introduction.** The nonclassical orthogonal polynomials which are eigenfunctions of a fourth order differential operator were classified by H. L. Krall [6], [7]. These polynomials were described in more details by A. M. Krall [5]. The corresponding weight functions are special cases of the classical weight functions together with a delta function at the end point(s) of the interval of orthogonality. A number of A. M. Krall's results can be obtained in a more satisfactory way:

(a) Jacobi, Legendre and Laguerre type polynomials are connected with each other by quadratic transformations and a limit formula.

(b) The power series for the Jacobi type polynomials is of ${}_3F_2$-type.

(c) There is a pair of second order differential operators not depending on n which connect the Jacobi polynomials $P_n^{(\alpha,0)}(2x-1)$ and the Jacobi type polynomials $S_n(x)$. Combination of these two differentiation formulas yields the fourth order equation for $S_n(x)$.

It is the first purpose of the present paper to make these comments to [5]. The second purpose is to describe a more general class of Jacobi type polynomials, with weight function $(1-x)^\alpha(1+x)^\beta$ + linear combination of $\delta(x+1)$ and $\delta(x-1)$. They can be expressed in terms of Jacobi polynomials as $((a_nx+b_n)d/dx+c_n)P_n^{(\alpha,\beta)}(x)$ for certain coefficients a_n, b_n, c_n and their power series in $\frac{1}{2}(1-x)$ is of ${}_4F_3$ type. Finally, they satisfy a second order differential equation with polynomial coefficients depending on n, but of bounded degree, thus generalizing the known result for the Jacobi type polynomials $S_n(x)$ (cf. Littlejohn & Shore [9]) and providing further examples for the general theory

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of orthogonal polynomials with this property, cf. Atkinson & Everitt [1], Hahn [4].

There are two further motivations for studying this class of orthogonal polynomials. First, as pointed out by Nikishin [11], any new set of orthogonal polynomials for which explicit expressions are available, is welcome because it provides a testing ground for the general theory of orthogonal polynomials. Second, orthogonal polynomials expressible in terms of certain hypergeometric functions may yield possibly new formulas for these hypergeometric functions. I want to acknowledge useful comments by D. Stanton on this topic, which led to section 8 of this paper.

1. Jacobi polynomials. We summarize the properties of Jacobi polynomials we need, cf. [3, §10.8].

Let $\alpha, \beta > -1$. *Jacobi polynomials* $P_n^{(\alpha,\beta)}(x)$ are orthogonal polynomials on the interval $[-1, 1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ and with the normalization

$$P_n^{(\alpha,\beta)}(1) = (\alpha+1)_n/n!. \tag{1.1}$$

Symmetry properties:

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x). \tag{1.2}$$

Differentiation formula:

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{1.3}$$

Rodrigues formula:

$$(-1)^n 2^n n!\,(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = (d/dx)^n((1-x)^{n+\alpha}(1+x)^{n+\beta}). \tag{1.4}$$

Power series expansion:

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right) \\ &= \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_k k!} \left(\frac{1-x}{2}\right)^k. \end{aligned} \tag{1.5}$$

Laguerre polynomials:

$$L_n^\alpha(x) := \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-2\beta^{-1}x), \tag{1.6}$$

orthogonal on $[0,\infty)$ with respect to the weight function $e^{-x}x^\alpha$.
Differential equation:

$$\begin{aligned} &[(1-x^2)d^2/dx^2 + (\beta-\alpha-(\alpha+\beta+2)x)d/dx]P_n^{(\alpha,\beta)}(x) \\ &\qquad = -n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x). \end{aligned} \tag{1.7}$$

2. **Definition.** Fix $M, N \geq 0$ and $\alpha, \beta > -1$. For $n = 0, 1, 2, \ldots$ define

$$P_n^{\alpha,\beta,M,N}(x) := ((\alpha+\beta+1)_n/n!)^2[(\alpha+\beta+1)^{-1}(B_nM(1-x) - A_nN(1+x))d/dx + A_nB_n]P_n^{(\alpha,\beta)}(x), \tag{2.1}$$

where

$$A_n := \frac{(\alpha+1)_n n!}{(\beta+1)_n(\alpha+\beta+1)_n} + \frac{n(n+\alpha+\beta+1)M}{(\beta+1)(\alpha+\beta+1)}, \tag{2.2}$$

$$B_n := \frac{(\beta+1)_n n!}{(\alpha+1)_n(\alpha+\beta+1)_n} + \frac{n(n+\alpha+\beta+1)N}{(\alpha+1)(\alpha+\beta+1)}. \tag{2.3}$$

The case $\alpha+\beta+1=0$ must be understood by continuity in α, β. By using (1.1) and (1.3) we find

$$P_n^{\alpha,\beta,M,N}(1) = \frac{(\alpha+1)_n}{n!} + \frac{(\beta+1)_n(\alpha+\beta+2)_n nM}{n!\,n!\,(\beta+1)}. \tag{2.4}$$

From (1.2) we have the symmetry

$$P_n^{\alpha,\beta,M,N}(-x) = (-1)^n P_n^{\beta,\alpha,N,M}(x). \tag{2.5}$$

3. **Orthogonality.** Define the measure μ on $[-1, 1]$ by

$$\int_{-1}^{1} f(x)\,d\mu(x) := \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} f(x)(1-x)^\alpha(1+x)^\beta\,dx + Mf(-1) + Nf(1), \qquad f \in C([-1, 1]). \tag{3.1}$$

THEOREM 3.1. *The polynomials $P_n^{\alpha,\beta,M,N}(x)$ are orthogonal polynomials on the interval $[-1, 1]$ with respect to the measure μ and with the normalization* (2.4).

Proof. By (2.1) and (2.3), $P_n^{\alpha,\beta,M,N}(x)$ is a polynomial of degree $\leq n$, not identically zero.

In order to prove the orthogonality first assume $n \geq 2$. Observe that the polynomials $(1+x)^k(1-x)^{n-k-1}$ $(k = 0, 1, \ldots, n-1)$ form a basis for the space of polynomials of degree $\leq n-1$. If $1 \leq k \leq n-2$ then

$$\int_{-1}^{1} P_n^{\alpha,\beta,M,N}(x)(1-x)^{n-k-1}(1+x)^k\,d\mu(x) = 0$$

by integration by parts and the orthogonality property of Jacobi polynomials. Now consider $k = 0$:

$$I := \int_{-1}^{1} P_n^{\alpha,\beta,M,N}(x)(1-x)^{n-1}\,d\mu(x).$$

The continuous part of μ yields a contribution

$$I_1 := \frac{\Gamma(\alpha+\beta+1)(n+\alpha+\beta+1)B_n M((\alpha+\beta+1)_n)^2}{2^{\alpha+\beta+3-n}\Gamma(\alpha+1)\Gamma(\beta+1)(n!)^2} \times \int_{-1}^{1} P_{n-1}^{(\alpha+1,\beta+1)}(x)(1-x)^{\alpha+1}(1+x)^{\beta}\,dx,$$

where we used (1.3) and the orthogonality property of Jacobi polynomials. Now substitute (1.4), integrate by parts and evaluate the resulting beta integral:

$$I_1 = (-1)^{n-1}2^{n-1}(\alpha+1)_n B_n M(\alpha+\beta+1)_n/(n!)^2.$$

The discrete part of μ yields a contribution $-I_1$ to I (use (1.5), (1.2) and (1.1)) so $I=0$. The case $k=n-1$ follows from the case $k=0$ by (2.5).

Finally consider the case $n=1$. By (1.5) we have

$$P_1^{(\alpha,\beta)}(x) = (\alpha+1) - \tfrac{1}{2}(\alpha+\beta+2)(1-x),$$

so

$$P_1^{\alpha,\beta,M,N}(x) = -\tfrac{1}{2}(\alpha+1)(\alpha+\beta+1)B_1(1-x) + \tfrac{1}{2}(\beta+1)(\alpha+\beta+1)A_1(1+x).$$

Hence

$$\int_{-1}^{1} P_1^{\alpha,\beta,M,N}(x)\,d\mu(x) = 0$$

by evaluating the beta integrals. □

4. Special cases. Of course:

$$P_n^{\alpha,\beta,0,0}(x) = P_n^{(\alpha,\beta)}(x). \tag{4.1}$$

Next we have

$$P_n^{\alpha,\beta,M,0}(x) = \left[1 + \frac{M(\beta+1)_n(\alpha+\beta+1)_n}{(\alpha+1)_n n!\,(\alpha+\beta+1)}\left((1-x)\frac{d}{dx} + \frac{n(n+\alpha+\beta+1)}{\beta+1}\right)\right]P_n^{(\alpha,\beta)}(x), \tag{4.2}$$

$$S_n(x) = MP_n^{\alpha,0,(\alpha+1)/M,0}(2x-1) = ((1-x)d/dx + n(n+\alpha+1) + M)P_n^{(\alpha,0)}(2x-1), \tag{4.3}$$

where $S_n(x)$ are Krall's [5, §16,17] Jacobi type polynomials, orthogonal with respect to the measure $((1-x)^\alpha + M^{-1}\delta(x))\,dx$ on $[0, 1]$.

Furthermore,

$$P_n^{\alpha,\alpha,M,M}(x) = \left(1 + \frac{M(2\alpha+2)_n n}{(\alpha+1)n!}\right) \cdot \left[1 + \frac{M(2\alpha+1)_n}{n!\,(2\alpha+1)}\left(-2x\frac{d}{dx} + \frac{n(n+2\alpha+1)}{\alpha+1}\right)\right]P_n^{(\alpha,\alpha)}(x), \tag{4.4}$$

$$P_n^{(\alpha)}(x) = \frac{\alpha^2}{\alpha+\frac{1}{2}n(n+1)}P_n^{0,0,1/(2\alpha),1/(2\alpha)}(x) = (-x\,d/dx + \alpha + \tfrac{1}{2}n(n+1))P_n(x), \tag{4.5}$$

where $P_n^{(\alpha)}(x)$ are Krall's [5, §4.5] Legendre type polynomials, orthogonal with respect to the measure $\frac{1}{2}(\alpha+\delta(x-1)+\delta(x+1))\,dx$ on $[-1,1]$.

By using Theorem 3.1 we obtain the quadratic transformations

$$\text{(4.6)}\qquad \frac{P_{2n}^{\alpha,\alpha,M,M}(x)}{P_{2n}^{\alpha,\alpha,M,M}(1)}=\frac{P_n^{\alpha,-1/2,0,2M}(2x^2-1)}{P_n^{\alpha,-1/2,0,2M}(1)},$$

$$\text{(4.7)}\qquad \frac{P_{2n+1}^{\alpha,\alpha,M,M}(x)}{P_{2n+1}^{\alpha,\alpha,M,M}(1)}=\frac{xP_n^{\alpha,1/2,0,(4\alpha+6)M}(2x^2-1)}{P_n^{\alpha,1/2,0,(4\alpha+6)M}(1)}.$$

In particular, these formulas connect Krall's Legendre and Jacobi type polynomials with each other.

$$\text{(4.8)}\quad L_n^{\alpha,N}(x):=\lim_{\beta\to\infty}P_n^{\alpha,\beta,0,N}(1-2\beta^{-1}x)=\left[1+\frac{N(\alpha+1)_n}{n!}\left(\frac{d}{dx}+\frac{n}{\alpha+1}\right)\right]L_n(x),$$

orthogonal polynomials on the interval $[0,\infty)$ with respect to the measure $((\Gamma(\alpha+1))^{-1}e^{-x}x^\alpha+N\delta(x))\,dx$ on the interval $[0,\infty)$ and with the normalization $L_n^{\alpha,N}(0)=(\alpha+1)_n/n!$ (cf. (1.6), (2.5), (4.2) and Theorem 3.1).

$$\text{(4.9)}\qquad R_n(x)=RL_n^{0,R^{-1}}(x),$$

where $R_n(x)$ are Krall's [5, §10,11] Laguerre type polynomials, orthogonal with respect to the measure $(e^{-x}+R^{-1}\delta(x))\,dx$ on $[0,\infty)$.

5. Expression as hypergeometric series. By (1.5) and (2.1) we have

$$\frac{n!\,n!\,n!}{(\alpha+1)_n(\alpha+\beta+1)_n(\alpha+\beta+1)_n}P_n^{\alpha,\beta,M,N}(1-2x)$$
$$=[(\alpha+\beta+1)^{-1}(-B_nMx+A_nN(1-x))d/dx+A_nB_n]$$
$$\cdot\left(\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_k k!}x^k\right).$$

By straightforward calculations we obtain

$$\text{(5.1)}\quad \frac{P_n^{\alpha,\beta,M,N}(1-2x)}{P_n^{\alpha,\beta,M,N}(1)}=\frac{(\alpha+1)_n(\alpha+\beta+1)_n}{(\alpha+1)(\beta+1)_n n!\,A_n}$$
$$\cdot\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+2)_k k!}\Big[-MB_n(\alpha+\beta+1)^{-1}k^2$$
$$+(NA_n(\alpha+\beta+1)^{-1}\beta-MB_n(\alpha+\beta+1)^{-1}(\alpha+1)+A_nB_n)k$$
$$+\frac{(\alpha+1)(\beta+1)_n n!}{(\alpha+1)_n(\alpha+\beta+1)_n}A_n\Big]x^k.$$

For $M,N>0$ this becomes

$$\text{(5.2)}\qquad \frac{P_n^{\alpha,\beta,M,N}(1-2x)}{P_n^{\alpha,\beta,M,N}(1)}={}_4F_3\left(\begin{matrix}-n,n+\alpha+\beta+1,-a_n+1,b_n+1\\ \alpha+2,-a_n,b_n\end{matrix}\,\middle|\,x\right),$$

where $a_n > n$, $b_n > 0$ and

$$a_n b_n = \frac{(\alpha+1)(\alpha+\beta+1)(\beta+1)_n n!\, A_n}{(\alpha+1)_n(\alpha+\beta+1)_n M B_n},$$

$$a_n - b_n = \beta N M^{-1} A_n B_n^{-1} + (\alpha+\beta+1)M^{-1}A_n - \alpha - 1.$$

For $M=0, N\neq 0$:

$$\frac{P_n^{\alpha,\beta,0,N}(1-2x)}{P_n^{\alpha,\beta,0,N}(1)} = {}_3F_2\left(\begin{matrix}-n,\ n+\alpha+\beta+1,\ c_n+1\\ \alpha+2,\ c_n\end{matrix}\ \middle|\ x\right), \tag{5.3}$$

where

$$c_n = \frac{(\alpha+1)(\beta+1)_n n!}{(N(\alpha+\beta+1)^{-1}\beta + B_n)(\alpha+1)_n(\alpha+\beta+1)_n}.$$

For $N=0, M\neq 0$:

$$\frac{P_n^{\alpha,\beta,M,0}(1-2x)}{P_n^{\alpha,\beta,M,0}(1)} = {}_3F_2\left(\begin{matrix}-n,\ n+\alpha+\beta+1,\ -(\alpha+\beta+1)M^{-1}A_n+1\\ \alpha+1,\ -(\alpha+\beta+1)M^{-1}A_n\end{matrix}\ \middle|\ x\right). \tag{5.4}$$

Combination of (4.3), (2.5) and (5.3) yields Krall's power series expansion [5, §16]. Combination of (4.6), (4.7), (2.5) and (5.4) yields power series expansion in x for $P_n^{\alpha,\alpha,M,M}(x)$, cf. [5, §4].

6. Second order differential equations. In view of the observations in Atkinson & Everitt [1, §6] and the definition (3.1) of our orthogonality measure it is no surprise that the polynomials $P_n^{\alpha,\beta,M,N}$ will satisfy a linear second order ordinary differential equation with polynomial coefficients, n-dependent but of bounded degree. Hahn [4, §6] points out that, if $\{u_n\}$ and $\{y_n\}$ are systems of orthogonal polynomials and $u_n = y_n + q_n y_n'$ for certain first degree polynomials q_n, then the y_n's satisfy second order o.d.e.'s of the above type. Our relation (2.1) has this form, but here Hahn's observation yields nothing new, since the second order o.d.e. for the $P_n^{(\alpha,\beta)}$'s is already well-known (cf. (1.7)). However, we can prove:

PROPOSITION 6.1 *Let $\{u_n\}$ and $\{y_n\}$ be systems of orthogonal polynomials such that*

$$u_n = p_n y_n + q_n y_n', \tag{6.1}$$

$$y_n'' + \alpha_n y_n' + \beta_n y_n = 0, \tag{6.2}$$

where $p_n, q_n, \alpha_n, \beta_n$ are rational functions which are quotients of polynomials of bounded degree. Then

$$y_n = r_n u_n + s_n u_n', \tag{6.3}$$

$$u_n'' + \gamma_n u_n' + \delta_n u_n = 0, \tag{6.4}$$

for certain rational functions r_n, s_n, γ_n, δ_n which are quotients of polynomials of bounded degree.

Proof. Clearly, we only need to prove the proposition for sufficiently large n and under the assumption that q_n is not identically zero. Eliminate y_n' and y_n'' from (6.1), (6.2) and the equation obtained by differentiating (6.1) once. Then we obtain

$$(q_n(p_n'-\beta_n q_n)-p_n(p_n+q_n'-\alpha_n q_n))y_n=(-p_n+\alpha_n q_n-q_n')u_n+q_n u_n'.$$

Here the coefficient of y_n is not identically zero for sufficiently large n, because, otherwise, not all zeros of u_n would be simple, in contradiction to Szegö [12, Theorem 3.3.1]. This proves (6.3). Next eliminate y_n and y_n' from (6.1), (6.3) and the first derivative of (6.3). Then we obtain

$$s_n u_n''+(p_n s_n+q_n(r_n+s_n'))u_n'+(p_n r_n+q_n r_n'-1)u_n=0.$$

Since we assumed $q_n\neq 0$, we have $s_n\neq 0$. This proves (6.4). □

Now apply Prop. 6.1 to the case $y_n=P_n^{\alpha,\beta}$, $u_n=P_n^{\alpha,\beta,M,N}$. It follows from (2.1) and (1.7) that

$$(a_n(x)d/dx+b_n(x))P_n^{\alpha,\beta,M,N}(x)=((\alpha+\beta+1)_n/n!)^2c_n(x)P_n^{(\alpha,\beta)}(x), \tag{6.5}$$

where

$$a_n(x):=(B_nM-A_nN-(B_nM+A_nN)x)(1-x^2),$$

$$b_n(x):=(\alpha+\beta+1)(B_nM+A_nN+A_nB_n)x^2+2((\alpha+1)A_nN-(\beta+1)B_nM)x$$
$$+(\beta-\alpha+1)B_nM+(\alpha-\beta+1)A_nN-A_nB_n(\alpha+\beta+1),$$

$$c_n(x):=A_nB_nb_n(x)-n(n+\alpha+\beta+1)$$
$$\times(\alpha+\beta+1)^{-1}(B_nM-A_nN-(B_nM+A_nN)x)^2.$$

From (6.5) one can calculate the second order o.d.e. for $P_n^{\alpha,\beta,M,N}$. Littlejohn & Shore [9] derive special cases of this o.d.e. for the polynomials (4.3), (4.5), (4.9) in a different, more complicated way.

7. Fourth order differential equation for Krall's Jacobi type polynomials. Fix $\alpha>-1$ and $M>0$. Let $S_n(x)$ be defined by (4.3). Combination of (4.3) and (1.7) yields

$$S_n(x)=[x(x-1)d^2/dx^2+(\alpha+1)xd/dx+M]P_n^{(\alpha,0)}(2x-1). \tag{7.1}$$

Observe that, for arbitrary polynomials f, g we have

$$\int_0^1 g(x)[x(x-1)d^2/dx^2+(\alpha+1)xd/dx+M]f(x)((1-x)^\alpha+M^{-1}\delta(x))\,dx$$
$$=\int_0^1 f(x)[x(x-1)/d^2/dx^2+((\alpha+3)x-2)d/dx+M+\alpha+1]g(x)(1-x)^\alpha\,dx. \tag{7.2}$$

Formulas (7.1), (7.2) and the orthogonality properties of $S_n(x)$ and $P_n^{(\alpha,0)}(2x-1)$ imply:

$$(7.3)\qquad \begin{aligned}&((n+\alpha+1)(n+1)+M)(n(n+\alpha)+M)P_n^{(\alpha,0)}(2x-1)\\ &\qquad=\left[x(x-1)\frac{d^2}{dx^2}+((\alpha+3)x-2)\frac{d}{dx}+M+\alpha+1\right]S_n(x),\end{aligned}$$

where the coefficient of $P_n^{(\alpha,0)}(2x-1)$ is obtained by comparing the coefficients of x^n at both sides of (7.3). Combination of (7.1) and (7.3) yields.

THEOREM 7.1. *The polynomials $S_n(x)$ are eigenfunctions of a fourth order differential operator with polynomial coefficients not depending on n.*

A calculation leads to the explicit form of Krall's [5, §14] differential equation.

Recently Littlejohn [8], proved that the polynomials $P_n^{0,0,M,N}(x)$ (notation of the present paper) are eigenfunctions of a sixth order differential operator. The above techniques also apply to this case and would lead to an eighth order differential operator.

8. Quadratic transformations for hypergeometric functions. Many of the formulas for terminating ${}_2F_1$ hypergeometric functions can be derived from similar formulas for Jacobi polynomials (use (1.5)) obtained by properties of orthogonal polynomials. Similarly, results for the polynomials $P_n^{\alpha,\beta,M,N}$ obtained here can be translated in terms of hypergeometric functions of the form

$${}_4F_3\left(\begin{matrix}-n, b, \theta_1+1, \theta_2+1\\ c, \theta_1, \theta_2\end{matrix}\,\middle|\, x\right),\qquad n=0,1,2,\ldots$$

(use (5.2)). In particular, (4.6) and (4.7) will imply quadratic transformations for such ${}_4F_3$-functions. In this section we will give an independent derivation of these quadratic transformations, also in the nonterminating case.

Our starting point is

$$(8.1)\qquad \begin{aligned}&(1-x)^{-a}{}_4F_3\left(\begin{matrix}\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}, b, d+1\\ c, b+a+2-c, d\end{matrix}\,\middle|\, -\frac{4x}{(1-x)^2}\right)\\ &\quad={}_5F_4\left(\begin{matrix}a, 1+a-c, c-1-b, \theta_1+1, \theta_2+1\\ c, a+b+2-c, \theta_1, \theta_2\end{matrix}\,\middle|\, x\right),\end{aligned}$$

where

$$\theta_1+\theta_2=a,\ \theta_1\theta_2=\frac{d(1+a-c)(c-1-b)}{d-b}$$

(formula due to D. Stanton, private communication). For the proof expand the left hand side as a power series. On letting $b \to \infty$ in (8.1) we obtain

$$(8.2)\qquad \begin{aligned}(1-x)^{-a}{}_3F_2&\left(\begin{matrix}\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}, d+1\\ c, d\end{matrix}\;\middle|\; -\frac{4x}{(1-x)^2}\right)\\ &= {}_4F_3\left(\begin{matrix}a, 1+a-c, \theta_1+1, \theta_2+1\\ c, \theta_1, \theta_2\end{matrix}\;\middle|\; -x\right),\end{aligned}$$

where $\theta_1+\theta_2=a$, $\theta_1\theta_2=d(1+a-c)$. Observe that (8.1) tends to formula (22) in Niblett [10] as $b \to \infty$ and to formula 4.5 (1) in [2] as $d \to b$.

A linear transformation formula is given by

$$(8.3)\quad (1-x)^{-a}{}_4F_3\left(\begin{matrix}a, b, d+1, e+1\\ c, d, e\end{matrix}\;\middle|\; \frac{x}{x-1}\right) = {}_4F_3\left(\begin{matrix}a, c-b-2, \theta_1+1, \theta_2+1\\ c, \theta_1, \theta_2\end{matrix}\;\middle|\; x\right),$$

where

$$\theta_1+\theta_2=\frac{(d+e)(b^2-bc+2b)+de(-2b+2c-3)+(1-c)b}{(d-b)(e-b)},$$

$$\theta_1\theta_2=\frac{de(c-b-1)(c-b-2)}{(d-b)(e-b)}.$$

For the proof again expand the left hand side. A limit case of (8.3) is

$$(8.4)\qquad \begin{aligned}(1-x)^{-a}{}_3F_2&\left(\begin{matrix}a, b, d+1\\ c, d\end{matrix}\;\middle|\; \frac{x}{x-1}\right)\\ &= {}_3F_2\left(\begin{matrix}a, c-b-1, d(c-b-1)(d-b)^{-1}+1\\ c, d(c-b-1)(d-b)^{-1}\end{matrix}\;\middle|\; x\right).\end{aligned}$$

Substitution of (8.3) and (8.4) into (8.2) yields the two formulas

$$(8.5)\qquad {}_3F_2\left(\begin{matrix}a, b, d+1\\ a+b+\frac{3}{2}, d\end{matrix}\;\middle|\; 4x(1-x)\right) = {}_4F_3\left(\begin{matrix}2a, 2b, \theta_1+1, \theta_2+1\\ a+b+\frac{3}{2}, \theta_1, \theta_2\end{matrix}\;\middle|\; x\right),$$

where

$$\theta_1+\theta_2=\frac{4ab+a+b+d+\frac{1}{2}}{a+b-d+\frac{1}{2}},$$

$$\theta_1\theta_2=\frac{4(a+\frac{1}{2})(b+\frac{1}{2})d}{a+b-d+\frac{1}{2}},$$

$$(8.6)\quad (1-2x){}_3F_2\left(\begin{matrix}a, b, d+1\\ d+b+\frac{1}{2}, d\end{matrix}\;\middle|\; 4x(1-x)\right) = {}_4F_3\left(\begin{matrix}2a-1, 2b-1, \theta_1+1, \theta_2+1\\ a+b+\frac{1}{2}, \theta_1, \theta_2\end{matrix}\;\middle|\; x\right),$$

where

$$\theta_1+\theta_2=\frac{4ab-a-b-d+\frac{1}{2}}{a+b-d-\frac{1}{2}}$$

$$\theta_1\theta_2=\frac{4(a-\frac{1}{2})(b-\frac{1}{2})d}{a+b-d-\frac{1}{2}}.$$

Formulas (8.5) and (8.6) imply (4.6) and (4.7), respectively.

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